

Biinvariant functions on the group of transformations leaving a measure quasiinvariant

NERETIN YU.A.¹

Let Gms be the group of transformations of a Lebesgue space leaving the measure quasiinvariant, let Ams be its subgroup consisting of transformations preserving the measure. We describe canonical forms of double cosets of Gms by the subgroup Ams and show that all continuous Ams -biinvariant functions on Gms are functionals on of the distribution of a Radon–Nikodym derivative.

1 Statements

1.1. The group Gms . By \mathbb{R}^\times we denote the multiplicative group of positive reals. By t we denote the coordinate on \mathbb{R}^\times .

Let M be a Lebesgue space (see [1]) with a continuous probabilistic measure μ (recall that any such space is equivalent to the segment $[0, 1]$). Denote by $Ams = Ams(M)$ the group of all transformations (defined up to a.s.) preserving the measure μ . By $Gms = Gms(M)$ we denote the group of transformations (defined up to a.s.) leaving the measure μ quasiinvariant.

The group Ams was widely discussed in connection with ergodic theory, the group Gms , which is a topic of the present note, only occasionally was mentioned in the literature. However, it is an interesting object from the point of view of representations of infinite-dimensional groups (“large groups” in the terminology of A.M.Vershik), see [2], [3].

1.2. The topology on Gms . A separable topology on Gms was defined in [4] 17.46, [5], [6], §4.5 by different ways. One of the purposes of the present note is two show that these ways are equivalent.

The first way is following. Let $A, B \subset M$ be measurable subsets. For $g \in Gms$ we define the distribution

$$\varkappa[g; A, B]$$

of the Radon–Nikodym derivative g' on the set $A \cap g^{-1}(B)$. We say that a sequence $g_j \in Gms$ converges to g , if for any measurable sets A, B we have the following weak convergences of measures on \mathbb{R}^\times

$$\varkappa[g_j; A, B] \rightarrow \varkappa[g; A, B], \quad t\varkappa[g_j; A, B] \rightarrow t\varkappa[g; A, B]. \quad (1.1)$$

REMARK 1. Point out evident identities:

$$\int_{\mathbb{R}^\times} \varkappa[g; A, M](t) = \mu(A), \quad \int_{\mathbb{R}^\times} t \varkappa[g; A, M](t) = \mu(gA). \quad (1.2)$$

REMARK 2. Consider a measurable finite partition

$$\mathfrak{h} : M = M^1 \cup M^2 \cup \dots$$

¹Supported by the grant FWF, P25142.

of the space M . This gives us a matrix $S_{\alpha\beta}[g; \mathfrak{h}] := \varkappa[g; M^\alpha, M^\beta]$, composed of measures on \mathbb{R}^\times . If a partition \mathfrak{k} is a refinement of \mathfrak{h} , we write $\mathfrak{h} \preceq \mathfrak{k}$. Consider a sequence of partitions $\mathfrak{h}_1 \preceq \mathfrak{h}_2 \preceq \dots$, generating the σ -algebra of the space² M . A convergence $g_j \rightarrow g$ is equivalent to an element-wise convergence in the sense (1.1) of all matrices $S[g_j; \mathfrak{h}_n] \rightarrow S[g; \mathfrak{h}_n]$.

Proposition 1.1 *The group Gms is a Polish group with respect to this topology, i.e., Gms is a separable topological group complete with respect to the two-side uniform structure and homeomorphic to a complete metric space³.*

Let $1 \leq p \leq \infty$, $s \in \mathbb{R}$. The group Gms acts in the space $L^p(M)$ by isometric transformations according the formula

$$T_{1/p+is}f(x) = f(g(x))g'(x)^{1/p+is}.$$

On the space $\mathcal{B}(V)$ of operators of a Banach space V we define in the usual way (see, e.g., [7], VI.1) the strong and weak topologies. Also, on the set $\mathcal{GL}(V)$ of invertible operators we introduce a *bi-strong* topology, A_j converges to A , if $A_j \rightarrow A$ and $A_j^{-1} \rightarrow A^{-1}$ strongly. The embedding $T_{1/p+is} : \text{Gms} \rightarrow \mathcal{B}(L^p)$ induces a certain topology on Gms from any operator topology on $\mathcal{B}(V)$ or $\mathcal{GL}(V)$.

Proposition 1.2 a) *Let $1 < p < \infty$, $s \in \mathbb{R}$. A topology on Gms induced from any of three topologies (strong, weak, bi-strong) coincides with the topology defined above.*

b) *Let $p = 1$, $s \in \mathbb{R}$. A topology on Gms induced from strong or bi-strong topology coincides with the topology defined above.*

c) *Let $1 \leq p < \infty$, $s \in \mathbb{R}$. Then the image of Gms in $\mathcal{GL}(L^p(M))$ is closed in the bi-strong topology.*

Point out that the coincidence of topologies is not surprising. It is known that two different Polish topologies on a group can not determine the same Borel structure, see [4], 12.24. There are also theorems about automatic continuity of homomorphisms, see [4], 9.10,

1.3. Double cosets $\text{Ams} \backslash \text{Gms} / \text{Ams}$. Canonical forms. We reformulate the problem of description of double cosets $\text{Ams} \backslash \text{Gms} / \text{Ams}$ in the following way. Let (P, π) , (R, ρ) be Lebesgue spaces with continuous probabilistic measures. Denote by $\text{Gms}(P, R)$ the space of all bijections $g : P \rightarrow R$ (defined up to a.s.),

²As \mathfrak{h}_n we can take a partition of the segment $M = [0, 1]$ into 2^n pieces of type $[k2^{-n}, (k+1)2^{-n})$

³A metric is compatible with the topology of the group, but not with its algebraic structure; in particular a metric is not assumed to be invariant. A completeness of a group in the sense of two-side uniform structure (in Raikov's sense [8]) is defined (for metrizable groups) in the following way. Let double sequences $g_i g_j^{-1}$ and $g_i^{-1} g_j$ converge to 1 as $i, j \rightarrow \infty$. Then g_i has a limit in the group. This definition is not equivalent to the definition of Bourbaki [9], III.3.3, who requires a completeness with respect to both one-side uniform structures. The group Gms is not complete in the sense of Bourbaki.

such that images and preimages of sets of zero measure have zero measure. We wish to describe such bijections up to the equivalence

$$g \sim u \cdot g \cdot v, \quad \text{where } v \in \text{Ams}(P), u \in \text{Ams}(R) \quad (1.3)$$

(clearly, such classes are in-to-one correspondence with double cosets $\text{Ams} \setminus \text{Gms}/\text{Ams}$).

Lemma 1.3 *Two elements $g_1, g_2 \in \text{Gms}(P, R)$ are contained in one class if and only if the Radon–Nikodym derivatives $g'_1, g'_2 : P \rightarrow \mathbb{R}$ are equivalent with respect to the action of the group $\text{Ams}(P)$, i.e., $g'_2(m) = g'_1(hm)$, where h is an element of $\text{Ams}(P)$.*

An evident invariant of this action is the distribution ν of the Radon–Nikodym derivative g' of the map g ,

$$\int_{\mathbb{R}^\times} d\nu(t) = 1, \quad \int_{\mathbb{R}^\times} t d\nu(t) = 1. \quad (1.4)$$

This invariant is not exhaust, the problem is reduced to the Rokhlin theorem [10] on metric classification of functions, see discussion below, §3.2. The final answer is following.

Consider a countable number of copies $\mathbb{R}_1^\times, \mathbb{R}_2^\times, \dots$ of half-line \mathbb{R}^\times . Consider one more more copy \mathbb{R}_∞^\times . Consider the disjoint union

$$\mathcal{L} := \mathbb{R}_1^\times \amalg \mathbb{R}_2^\times \amalg \mathbb{R}_3^\times \amalg \dots \amalg (\mathbb{R}_\infty^\times \times [0, 1]).$$

Let $\nu_1, \nu_2, \dots, \nu_\infty$ be a family of measure on \mathbb{R}^\times satisfying the following conditions

1. ν_1, ν_2, \dots are continuous (but ν_∞ admits atoms).
2. $\nu_1 \geq \nu_2 \geq \dots$
3. The measure $\nu := \nu_1 + \nu_2 + \dots + \nu_\infty$ satisfies (1.4).

Equip each \mathbb{R}_j^\times with the measure ν_j , equip $\mathbb{R}_\infty^\times \times [0, 1]$ with the measure $\nu_\infty \times dx$, where dx is the Lebesgue measure on the segment. Denote the resulting measure space by $\mathcal{L}[\nu_1, \nu_2, \dots; \nu_\infty]$.

Consider the same measure on \mathcal{L} multiplied by t , we denote the resulting measure space by $\mathcal{L}_*[\nu_1, \nu_2, \dots; \nu_\infty]$.

Consider the identity map

$$\text{id} : \mathcal{L}[\nu_1, \nu_2, \dots; \nu_\infty] \rightarrow \mathcal{L}_*[\nu_1, \nu_2, \dots; \nu_\infty] \quad (1.5)$$

Evidently, the distribution of the Radon–Nikodym derivative of the map id coincides with ν .

Proposition 1.4 *Any equivalence class (1.3) contains a unique representative of the type (1.5).*

Denote the double coset containing this representative by $S[\nu_1, \nu_2, \dots; \nu_\infty]$.

1.4. On closures of double cosets.

Theorem 1.5 *Let a measure ν on \mathbb{R}^\times satisfies (1.4), let $\nu = \nu^c + \nu^d$ be its decomposition into continuous and discrete parts. Then the closure of the double coset $S[\nu^c, 0, 0, \dots; \nu^d]$ contains all double cosets $S[\nu_1, \nu_2, \nu_3, \dots; \nu_\infty^c + \nu_d]$ with $\nu_1 + \nu_2 + \dots + \nu_\infty^c = \nu^c$.*

1.5. Hausdorff quotient. Consider the space \mathcal{M} of all measures ν on \mathbb{R}^\times satisfying (1.4). Say that $\nu^j \in \mathcal{M}$ converges to ν if $\nu^j \rightarrow \nu$ and $t\nu^j \rightarrow t\nu$ weakly.

Consider a map $\Phi : \text{Gms} \rightarrow \mathcal{M}$ that for any g assigns the distribution of its Radon–Nikodym derivative (i.e., $\Phi(g) = \varkappa[g; M, M]$). In virtue of Theorem 1.5, preimages of points $\nu \in \mathcal{M}$ are closures of double cosets $S[\nu^c, 0, 0, \dots; \nu^d]$.

Theorem 1.6 *Let f be a continuous map of Gms to a metric space T , moreover, f let be constant on double cosets. Then f has the form $f = q \circ \Phi$, where $q : \mathcal{M} \rightarrow T$ is a continuous map.*

1.6. A continuous section $\mathcal{M} \rightarrow \text{Gms}$. We say that a function $h : [0, 1] \rightarrow [0, 1]$ is contained in the class \mathcal{G} , if

- h is downward convex;
- $h(0) = 0$, $h(1) = 1$, and $h(x) > 0$ for $x > 0$.

Any such function is an element of the group $\text{Gms}([0, 1])$.

Proposition 1.7 *Let $\nu \in \mathcal{M}$. Then there is a unique function $\psi : [0, 1] \rightarrow [0, 1]$ of the class \mathcal{G} such that the distribution of the derivative ψ' is ν . Moreover, the map $\nu \mapsto \psi$ is a continuous map $\mathcal{M} \rightarrow \text{Gms}$.*

1.7. A more general statement. Consider a finite or countable measurable partition of our measure space $M = \coprod_j M_j$. Denote by K the direct product $K = \text{Ams}(M_1) \times \text{Ams}(M_2) \times \dots$. Consider the double cosets $K \setminus \text{Gms}/K$. Assign to each $g \in \text{Gms}$ the matrix $\varkappa_{ij} = \varkappa[g; M_i, M_j]$ composed of measures on \mathbb{R}^\times . Denote by \mathcal{S} the set of matrices that can be obtained in this way, i.e.,

$$\sum_j \int_{\mathbb{R}^\times} d\varkappa_{ij}(t) = \mu(M_i), \quad \sum_i \int_{\mathbb{R}^\times} t \varkappa_{ij}(t) = \mu(M_j).$$

Equip \mathcal{S} with element-wise convergence (1.1). Denote by Ψ the natural map $\text{Gms} \rightarrow \mathcal{S}$.

Theorem 1.8 *Let f be a continuous map from Gms to a metric space T . Then there exists a continuous map $q : \mathcal{S} \rightarrow T$, such that $f = q \circ \Psi$.*

Point out that this statement was actually used in [5], [2].

1.8. The structure of the note. The statements about topology on Gms are proved in §2, about double cosets in §3. Theorem 1.6 follows from Theorem 1.5. However, as the referee pointed out, the first statement is simpler than the second (and it is more important). Therefore in the beginning of §3 we present a separate proof of Theorem 1.6.

2 The topology on the group Gms

Below we prove Propositions 1.1 and 1.2. The main auxiliary statement is Lemma 2.4. The remaining lemmas are proved in a straightforward way.

Notation:

- δ_a is an probabilistic atomic measure \mathbb{R}^\times supported by a point a .
- $\{\cdot, \cdot\}_{pq}$ is the natural pairing of L^p and L^q , where $1/p + 1/q = 1$;
- χ_A is the indicator function of a set $A \subset M$, i.e., $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \notin A$.

2.1. Preliminary remarks on the spaces L^p .

1) Recall (see [11], §3.3) that for $p \neq 2$ the group of isometries $\text{Isom}(L^p(M))$ of the space $L^p(M)$ consists of operators of the form

$$R(g, \sigma)f(x) = \sigma(x)f(g(x))g'(x)^{1/p}, \quad (2.1)$$

where $g \in \text{Gms}$, and $\sigma : M \rightarrow \mathbb{C}$ is a function whose absolute value equals 1.

2) For $1 < p < \infty$ the space L^p is uniformly convex (see [12], §26.7), therefore the restrictions of the strong and weak topologies to the unit sphere coincide. Therefore on the group of isometries $\text{Isom}(L^p(M))$ the weak and strong operator topologies coincide.

3) Recall that for separable Banach spaces (in particular, for L^p with $p \neq \infty$) the group of all isometries equipped with bi-strong topology is a Polish group, see [4], 9.B9.

2.2. Preliminary remarks on the group Gms.

1) *The invariance of the topology.* Equip Gms with topology from Subsection 1.2. The product in Gms is separately continuous (this is a special case of Theorem 5.9 from [13]). In particular, this implies that the topology on Gms is invariant with respect to left and right shifts.

The map $g \mapsto g^{-1}$ is continuous. Indeed,

$$\varkappa[g^{-1}; B, A](t) = t^{-1}\varkappa[g; A, B](t^{-1}),$$

and this map transpose the convergences (1.1).

2) *Separability of Gms.* For a measure $\varkappa[g; A, B]$ consider the *characteristic function*

$$\chi(z) = \int_{\mathbb{R}^\times} t^z d\varkappa[g; A, B](t), \quad (2.2)$$

continuous in the strip $0 \leq \text{Re } z \leq 1$ and holomorphic in the open strip. The convergence of measures \varkappa is equivalent to point-wise convergence of characteristic functions uniform in each rectangle

$$0 \leq \text{Re } z \leq 1, \quad -N \leq \text{Im } z \leq N,$$

[13], Propositions 4.4-4.5. This convergence is separable. Next, by Remark 2 of §1.1, it suffices to verify the convergence of measures $\varkappa[g_j; A, B] \rightarrow \varkappa[g; A, B]$ for an appropriate countable set of pairs measurable subsets (A, B) .

3) *The action on Boolean algebra of sets.*

Lemma 2.1 *Let $g_j \rightarrow g$ in Gms. Then for any measurable set $A \subset M$ we have*

$$\mu(g_j A \triangle g A) \rightarrow 0. \quad (2.3)$$

PROOF. By the invariance of the topology it suffices to consider $g = 1$. Then

$$\begin{aligned} \mu(g_j A \cap A) &= \int_{\mathbb{R}^\times} d\mathcal{K}[g_j^{-1}; A; A](t) \rightarrow \int_{\mathbb{R}^\times} d\mathcal{K}[1; A; A](t) = \int_{\mathbb{R}^\times} \mu(A) \delta_0(t) = \mu(A); \\ \mu(g_j A) &= \int_{\mathbb{R}^\times} t d\mathcal{K}[g_j; A; M] \rightarrow \int_{\mathbb{R}^\times} t d\mathcal{K}[1; A; M] = \int_{\mathbb{R}^\times} \mu(A) t \delta_0(t) = \mu(A). \end{aligned}$$

Comparing two rows we get the desired statement. \square

REMARK. The opposite is false. Let $M = [0, 1]$,

$$g_j(x) = x + \frac{1}{2\pi n} \sin(2\pi n x).$$

Then for any $A \subset [0, 1]$ we have $\mu(g_j(A) \triangle A) \rightarrow \mu(A)$. But there is no convergence $g_j \rightarrow 1$ in Gms; $T_1(g_j)$ converges weakly to 1 in L^1 , but there is no strong convergence. \boxtimes

4) *The continuity of representations $T_{1/p+is}$.*

Lemma 2.2 *For $p < \infty$ the homomorphisms $T_{1/p+is} : \text{Gms} \rightarrow \text{Isom}(L^p)$ are continuous with respect to the weak topology $\text{Isom}(L^p)$.*

PROOF. Let $g_j \rightarrow g$ in Gms. Consider 'matrix elements'

$$\{T_{1/p+is}(g_j) \chi_A, \chi_B\}_{pq} = \int_{A \cap g_j^{-1} B} g'_j(x)^{1/p+is} d\mu(x) = \int_{\mathbb{R}^\times} t^{1/p+is} d\mathcal{K}[g_j; A, B](t)$$

Weak convergence of measures (1.1) implies the convergence of characteristic functions (2.2), our expression tends to

$$\int_{\mathbb{R}^\times} t^{1/p+is} d\mathcal{K}[g; A, B](t) = \int_{A \cap g^{-1} B} g'(x)^{1/p+is} d\mu(x) = \{T_{1/p+is}(g) \chi_A, \chi_B\}_{pq},$$

as required. \square

Thus, for $1 < p < \infty$ the maps $T_{1/p+is} : \text{Gms} \rightarrow \text{Isom}(L^p)$ are continuous with respect to the strong (=weak) topology. Keeping in mind the continuity of the map $g \mapsto g^{-1}$, we get that the maps $T_{1/p+is}$ are continuous with respect to the bi-strong topology.

The case L^1 must be considered separately.

Lemma 2.3 *Let $g_j \rightarrow g$. Then $T_{1+is}(g_j) \in \text{Isom}(L^1)$ strongly converges to $T_{1+is}(g)$.*

PROOF. Without loss of generality, we can set $g = 1$. It suffices to verify the convergence $\|T_{1+is}(g_j)\chi_A - \chi_A\| \rightarrow 0$ for any measurable A . This equals

$$\begin{aligned} & \int_M |\chi_A(g_j x) g'(x)^{1+is} - \chi_A(x)| d\mu(x) = \\ &= \int_{A \cap g_j^{-1} A} |g'(x)^{1+is} - 1| d\mu(x) + \int_{A \setminus g_j^{-1} A} d\mu(x) + \int_{g_j^{-1} A \setminus A} g'(x) d\mu(x) = \\ &= \int_{\mathbb{R}^\times} |t^{1+is} - 1| d\mathcal{K}[g_j; A, A](t) + \mu(A \setminus g_j^{-1} A) + \mu(A \setminus g_j A). \end{aligned} \quad (2.4)$$

The second and the third summands tend to 0 by Lemma 2.1, measures $\mathcal{K}[\dots]$ and $t\mathcal{K}[\dots]$ converge weakly to $\mu(A)\delta_0$, therefore the integral tends to 0. \square

2.3. The coincidence of topologies and the continuity of the multiplication.

Lemma 2.4 *Let $1 < p < \infty$. Let $T_{1/p+is}(g_j)$ weakly converge to 1 in $\text{Isom}(L_p)$. Then g_j converges to 1 in Gms .*

PROOF. *Step 1.* Now it will be proved that g'_j converges to 1 in $L^1(M)$. For this purpose, we notice that the following sequence of matrix elements must converge to 1:

$$\{T_{1/p+is}(g_j) 1, 1\}_{pq} = \int_M g'_j(x)^{1/p+is} d\mu(x) = \int_{\mathbb{R}^\times} t^{1/p+is} d\mathcal{K}[g_j; M, M](t). \quad (2.5)$$

Estimate the integrand:

$$\text{Re } t^{1/p+is} \leq t^{1/p} \leq \frac{1}{q} + \frac{t}{p}.$$

The second inequality means that the graph of upward convex function is lower than the tangent line at $t = 1$. From another hand:

$$\begin{aligned} & \int_{\mathbb{R}^\times} \left(\frac{1}{q} + \frac{t}{p} \right) d\mathcal{K}[g_j; M, M](t) = \\ &= \frac{1}{q} \int_{\mathbb{R}^\times} d\mathcal{K}[g_j; M, M](t) + \frac{1}{p} \int_{\mathbb{R}^\times} t d\mathcal{K}[g_j; M, M](t) = \frac{1}{q} + \frac{1}{p} = 1. \end{aligned}$$

Look to a deviation of integral (2.5) from 1. The same reasoning with tangent line allows to estimate the difference $\frac{1}{q} + \frac{t}{p} - t^{1/p}$. For any $\varepsilon > 0$ there is $\sigma > 0$ such that

$$\frac{1}{q} + \frac{t}{p} - t^{1/p} > \begin{cases} \sigma & \text{for } t < 1 - \varepsilon; \\ \sigma t & \text{for } t > 1 + \varepsilon. \end{cases}$$

Therefore

$$1 - \text{Re}\{T_{1/p+is}(g_j) 1, 1\}_{pq} > \sigma \int_0^{1-\varepsilon} d\mathcal{K}[g_j; M, M](t) + \sigma \int_{1+\varepsilon}^{\infty} t d\mathcal{K}[g_j; M, M](t).$$

This must tend to 0, therefore $\varkappa[g_j; M, M]$ and $t \cdot \varkappa[g_j; M, M]$ tend to δ_0 weakly. This implies the convergence $g'_j \rightarrow 1$ in the sense of L^1 .

The remaining part of the proof is more-or-less automatic.

Step 2. Let z be contained in the strip $0 \leq \operatorname{Re} z \leq 1$. Let us show that $(g'_j)^z$ tends to 1 in the sense of L^1 . Let $\|g' - 1\|_{L^1(M)} < \varepsilon$. Then there is an uniform with respect to g estimate $\|(g')^z - 1\|_{L^1(M)} < \psi_z(\varepsilon)$, where $\psi_z(\varepsilon)$ tends to 0 as ε tends to 0. For this aim it is sufficient to notice that

$$|a^z - 1| < \begin{cases} |z|(a-1) & \text{for } a > 1; \\ |z|2^{-\operatorname{Re} z+1}|a-1| & \text{for } 1/2 \leq a \leq 1; \\ 2 & \text{for } 0 < a < 1/2, \end{cases}$$

moreover, $g' < 1/2$ can be only on the set of measure $\leq 2\varepsilon$.

In particular, for any subset $C \subset M$ we have

$$\left| \int_C g'(x)^z dx - \mu(C) \right| \leq \psi_z(\varepsilon). \quad (2.6)$$

STEP 3. Now we use convergence of matrix elements:

$$\{T_{1/p+is}(g_j)\chi_A, \chi_B\}_{pq} = \int_{A \cap g_j^{-1}B} g'_j(x)^{1/p+is} d\mu(x) \rightarrow \{\chi_A, \chi_B\}_{pq} = \mu(A \cap B).$$

By (2.6), we have convergence

$$\int_{A \cap g_j^{-1}B} g'_j(x)^{1/p+is} d\mu(x) - \mu(A \cap g_j^{-1}B) \rightarrow 0.$$

Comparing two last convergences we get $\mu(A \cap g_j^{-1}B) \rightarrow \mu(A \cap B)$.

Step 4. By the convergence $(g'_j)^z$ in $L^1(M)$, we have

$$\int t^z d\varkappa[g_j; A, B](t) = \int_{A \cap g_j^{-1}B} g'(x)^z dx \rightarrow \mu(A \cap B)$$

for each z ; the point-wise convergence of characteristic functions implies weak converges (1.1) of measures (see, [13]), in our case, to $\mu(A \cap B)\delta_0$. \square

Thus the topology on Gms is induced from the strong operator topology of the spaces L^p . In separable Banach spaces the multiplication is continuous in the strong topology on bounded sets. Therefore, the multiplication in Gms is continuous.

Lemma 2.5 *Let operators $T_{1+is}(g_j)$ converge to 1 in the strong operator topology of spaces L^1 . Then $g_j \rightarrow 1$ in Gms.*

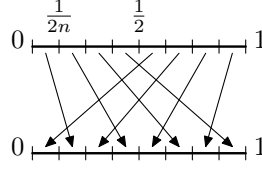


Figure 1: Reference to Example 2.5.

PROOF. In (2.4) the first row must tend to zero. Therefore all summands of the last row tend 0, in particular the first one. This implies weak convergences of measures $\varkappa[g_j; A, A]$ and $t\varkappa[g_j; A, A]$ to $\mu(A)\delta_1$. Comparing this with (1.2), we get convergences $\varkappa[g_j; A, M \setminus A]$ and $t\varkappa[g_j; A, M \setminus A]$ to 0. Now it is easy to derive the convergence of $g_j \rightarrow g$ in Gms. \square

2.4. The completeness of Gms. The group of isometries of a separable Banach space is a Polish group with respect to the bi-strong topology ([4], 9.3.9). Let $p \neq 1, 2, \infty$, $s = 0$. Then the isometries $T_{1/p}(g)$ are precisely isometries (2.1) that send the cone of non-negative functions to itself. Obviously, the set of operators sending this cone to itself is weakly closed. Therefore, Gms is a closed subgroup in the group of all isometries and therefore it is complete.

2.5. Bi-strong closeness of the image. The group Gms is closed in the group $\text{Isom}(L^p)$, since it is complete with respect of the induced topology.

It is noteworthy that the group $\text{Isom}(L^p)$ is not strongly closed in the space of bounded operators in L^p . The images of the groups Ams and Gms also are not closed.

EXAMPLE. Let $p \neq \infty$. Consider an operator in L^p of the form

$$Rf(x) = \begin{cases} f(2x), & \text{for } 0 \leq x \leq 1/2; \\ f(2x-1), & \text{for } 1/2 < x \leq 1; \end{cases}$$

For any function f we have $\|Rf\| = \|f\|$. However, this operator is not invertible. For the sequence $g_n \in \text{Ams}$ from Fig. 1 we have the strong convergence $T_{1/p}(g_n)$ to R . \boxtimes

Weak closures for some subgroups Gms are discussed in [2], [3].

3 Double cosets

3.1. Proof of Theorem 1.6. Denote by $G^0 \subset \text{Gms}$ the group of transformations whose Radon–Nikodym derivative has only finite number of values. Obviously,

- The subgroup G^0 is dense in Gms.
- Double cosets $\text{Ams} \backslash G^0 / \text{Ams}$ are completely determined by the distribution of the Radon–Nikodym derivative.

Consider a measure $\varkappa \in \mathcal{M}$. Consider a sequence of discrete measures $\varkappa_N \in \mathcal{M}$ convergent to \varkappa and having the following property: Fix N and cut the semi-axis $t > 0$ into pieces of length 2^{-N} . For any $j \in \mathbb{N}$ we require the following coincidence of measures of semi-intervals

$$\int_{\frac{j-1}{2^N} < t \leq \frac{j}{2^N}} d\varkappa(t) = \int_{\frac{j-1}{2^N} < t \leq \frac{j}{2^N}} d\varkappa_N(t), \quad \int_{\frac{j-1}{2^N} < t \leq \frac{j}{2^N}} t \cdot d\varkappa(t) = \int_{\frac{j-1}{2^N} < t \leq \frac{j}{2^N}} t \cdot d\varkappa_N(t)$$

Consider $g \in \text{Gms}$ whose distribution of the Radon–Nikodym derivative equals \varkappa . Consider a sequence $g_N \in G^0$ convergent to g such that a distribution of the Radon–Nikodym derivative of g_N is \varkappa_N . For this, we fix N and for each j consider the subset $A_j \subset M$, where the Radon–Nikodym derivative satisfies

$$\frac{j-1}{2^N} < g'(x) \leq \frac{j}{2^N}.$$

Set $B_j = g(A)$. Consider an arbitrary map $g_N \in G^0$ such that g_N send A_j to B_j and the distribution of the Radon–Nikodym derivative of g_N coincides with the restriction of the measure \varkappa_N of the semi-interval $(\frac{j-1}{2^N} < t \leq \frac{j}{2^N}]$. It easy to see that the sequence g_N converges to g .

Now, let f be a continuous function on Gms constant on double cosets. g and $h \in \text{Gms}$ have same distribution of Radon–Nikodym derivatives. Then g_N and h_N are contained in the same double coset, wherefore $f(g_N) = f(h_N)$. By continuity of f we get $f(g) = f(h)$.

To avoid a proof of the continuity the map q (see the statement of the theorem), we refer to Proposition 1.7 (which is proved below independently of the previous considerations).

3.2. Proof of Proposition 1.4. Let $M \simeq [0, 1]$ be a Lebesgue space. Invariants of measurable functions $f : M \rightarrow \mathbb{R}$ with respect to the action of $\text{Ams}(M)$ were described by Rokhlin in [10]. To any function f he assigns its distribution function $F(y)$, i.e., the measure of the set $M_y \subset M$ determined by the inequality $f(x) < y$. Also he assigns to f a sequence of functions F_1, F_2, \dots , where $F_n(y)$ is the supremum of measures of all sets $A \subset M_y$, on which f takes each value $\leq n$ times. These data satisfy the following conditions:

- the function F satisfies the usual properties of distribution functions: F is a left-continuous non-decreasing function, $\lim_{y \rightarrow -\infty} f(y) = 0$, $\lim_{y \rightarrow +\infty} f(y) = 1$;
- F_n are non-decreasing functions;
- $0 \leq F_1(y) \leq F_2(y) \leq \dots \leq F(y)$;
- $F_k(y) - 2F_{k+1}(y) + F_{k+2}(y) \geq 0$ for all k .

According [10], a function f determined up to the action of the group Ams is uniquely defined by the invariants F_1, F_2, \dots, F . Moreover, for any collection of functions F_1, F_2, \dots, F with above listed properties there exists f , whose invariants coincide with F_1, F_2, \dots, F .

Now we will describe canonical forms of functions f under the action of the group Ams . Consider a collection of continuous measures $\nu_1 \leq \nu_2 \leq \dots$ on \mathbb{R}

and the measure ν_∞ on \mathbb{R} such that $\nu_1(\mathbb{R}) + \nu_2(\mathbb{R}) + \dots + \nu_\infty(\mathbb{R}) = 1$. Denote by t the coordinate on \mathbb{R} . Consider the disjoint union of the spaces with measures

$$\mathcal{L} = \left((\mathbb{R}, \nu_1) \amalg (\mathbb{R}, \nu_2) \amalg \dots \right) \amalg (\mathbb{R} \times [0, 1], \nu_\infty \times ds), \quad (3.1)$$

where ds is the Lebesgue measure on the segment $[0, 1]$. Consider the function f on \mathcal{L} that equals to t on each copy of \mathbb{R} and equals to t on $\mathbb{R} \times [0, 1]$.

The invariants of this function are

$$F_n(y) = \sum_{j \leq n} \nu_j(-\infty, y), \quad F(y) = \sum_{1 \leq j < \infty} \nu_j(-\infty, y) + \nu_\infty(-\infty, y)$$

It can be readily seen that measures $\nu_1, \nu_2, \dots, \nu_\infty$ admit a reconstruction from the invariants F_1, F_2, \dots, F . Moreover any admissible collection of invariants corresponds to a certain collection of measures $\nu_1, \nu_2, \dots, \nu_\infty$.

Now consider an element $g \in \text{Gms}(P, R)$. Reduce the derivative $g' : P \rightarrow \mathbb{R}^\times$ to the canonical form by a multiplication $g \mapsto gh$, where $h \in \text{Ams}$. Since $g'(x) > 0$, all the measures ν_j, ν are supported by the half-line $t > 0$. The integral of g' is 1, therefore

$$\sum_j \int t d\nu_j(t) + \int t d\nu_\infty(t) = 1. \quad (3.2)$$

Now we assume $P = \mathcal{L}$, see (3.1). Let \mathcal{L}_* be obtained from \mathcal{L} by a multiplication of the measure by t . In virtue of (3.2), this measure must be probabilistic. The map $g : \mathcal{L} \rightarrow R$ can be regarded as a map $g_* : \mathcal{L}_* \rightarrow R$. Since $g' = t$, for any measurable set $B \subset \mathcal{L}$ the measure of B in \mathcal{L}_* coincides with the measure $g(B)$. Therefore $g_* : \mathcal{L}_* \rightarrow R$ preserves measure.

Thus g is reduced to the canonical form.

3.3. Splitting of measures. We start a proof of Theorem 1.5. Modify the notation for $\mathcal{L}[\nu_1, \nu_2, \dots; \nu_\infty]$, $\mathcal{L}_*[\nu_1, \nu_2, \dots; \nu_\infty]$ and $S[\nu_1, \nu_2, \dots; \nu_\infty]$ from (1.3). Now it is convenient to reject the condition $\nu_1 \geq \nu_2 \geq \dots$. Also, we weaken condition (1.4) and set

$$\int_{\mathbb{R}^\times} d\nu(t) < \infty, \quad \int_{\mathbb{R}^\times} t d\nu(t) < \infty. \quad (3.3)$$

Let ν be a continuous measure on \mathbb{R}^\times satisfying (3.3). Consider the space $\mathcal{L}[\nu, 0, 0, \dots; 0]$. Represent ν as a sum $\nu = \nu_1 + \nu_2$.

Lemma 3.1 *The closure of the class $S[\nu, 0, 0, \dots; 0]$ contains $S[\nu_1, \nu_2, 0, \dots; 0]$.*

PROOF. Denote

$$\mathcal{L} := \mathcal{L}[\nu_1, 0, \dots; 0], \quad \mathcal{L}' := \mathcal{L}[\nu_1, \nu_2, 0, \dots; 0].$$

The same measure spaces with the measure multiplied by t we denote as $\mathcal{L}_*, \mathcal{L}'_*$. Now we will construct two sequences of measure preserving bijections

$$\varphi_n : \mathcal{L} \rightarrow \mathcal{L}', \quad \psi_n : \mathcal{L}_* \rightarrow \mathcal{L}'_*.$$

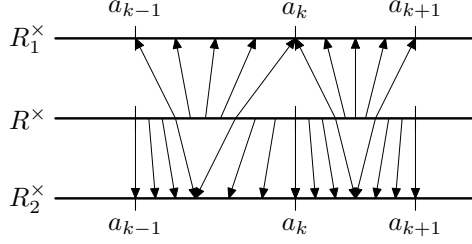


Figure 2: A reference to Lemma 3.1. The map φ_n .

Cut (\mathbb{R}^\times, ν) by 2^n intervals C_0, \dots, C_{2^n-1} by points

$$a_k = \frac{k2^{-n}}{1 - k2^{-n}}, \quad k = 1, 2, \dots, 2^n - 1.$$

Denote this partition⁴ by \mathfrak{h}_n .

Lemma 3.2 *The exists a sub-interval $B_k \subset C_k$ such that $\nu(B_k) = \nu_1(C_k)$, $(t \cdot \nu)(B_k) = (t \cdot \nu_1)(C_k)$.*

PROOF. We have $C_k = [a_k, a_{k+1}]$. Consider segments $[a_k, u]$, $[v, a_{k+1}] \subset [a_k, a_{k+1}]$ such that $\nu[a_k, u] = \nu[v, a_{k+1}] = \nu_1[C_k]$. For any $z \in [a_k, v]$ there exists $z^\circ \leq a_{k+1}$ such that $\nu[z, z^\circ] = \nu_1[C_k]$. It is easy to see that

$$(t \cdot \nu)[a_k, u] \leq \nu_1[C_k], \quad (t\nu)[v, a_{k+1}] \geq (t \cdot \nu_1)[C_k].$$

Form continuity reasoning there exists $[z, z^\circ]$ satisfying the desired property. \square

For each k consider arbitrary measure preserving maps

$$(B_k, \nu) \rightarrow (C_k, \nu_1), \quad (C_k \setminus B_k, \nu_2) \rightarrow (C_k, \nu_2).$$

This produces a map φ_n (see. Fig.2). To obtain ψ_n we take arbitrary measure preserving maps

$$(B_k, t \cdot \nu) \rightarrow (C_k, t \cdot \nu_1), \quad (C_k \setminus B_k, t \cdot \nu_2) \rightarrow (C_k, t \cdot \nu_2).$$

Consider a map

$$\theta_n : \psi_n \circ \text{id} \circ \varphi_n^{-1} : \mathcal{L}' \rightarrow \mathcal{L}'_*.$$

The space \mathcal{L}' consists of two copies $\mathbb{R}_1^\times, \mathbb{R}_2^\times$ of the half-line \mathbb{R}^\times , each copy is cutted into segments C_k . The map θ_n send each copy of a segment $C_k \subset \mathbb{R}_1^\times, C_k \subset \mathbb{R}_2^\times$ to itself, moreover the Radon–Nikodym derivative of θ_n takes values C_k in limits $[a_k, a_{k+1}]$.

It is easy to see that the sequence θ_n converges to the map $\text{id} : \mathcal{L}' \rightarrow \mathcal{L}'_*$. \square

⁴The only necessary for us property of partition is the following: a diameter of a partition on any finite interval $(0, M]$ tends to 0 as $n \rightarrow \infty$.

3.4. The spreading of measures. Denote

$$\mathcal{L} := \mathcal{L}[\nu, 0, \dots, 0], \quad \mathcal{L}'' = \mathcal{L}[0, 0, \dots; \nu].$$

Let \mathcal{L}_* , \mathcal{L}''_* be the same measure spaces with the measure multiplied by t . We construct a sequence of measure preserving bijections

$$\xi_n : \mathcal{L} \rightarrow \mathcal{L}'', \quad \zeta_n : \mathcal{L}_* \rightarrow \mathcal{L}''_*.$$

For this aim, consider the same partitions \mathfrak{h}_n of the space (\mathbb{R}^\times, ν) . Consider arbitrary measure preserving maps⁵

$$(C_k, \nu) \rightarrow (C_k \times [0, 1], \nu \times dx), \quad (C_k, t\nu) \rightarrow (C_k \times [0, 1], (t\nu) \times dx)$$

This gives us the maps ξ_n , ζ_n . Consider the map

$$v_n = \zeta_n \circ \text{id} \circ \xi_n^{-1} : \mathcal{L}'' \rightarrow \mathcal{L}''_*.$$

The map v_n sends each $C_k \times [0, 1]$ to itself, its Radon–Nikodym derivative on $C_k \times [0, 1]$ varies in the limits $[a_{k-1}, a_k]$. Passing to a limit as $n \rightarrow \infty$, we get the identity map $\mathcal{L}'' \rightarrow \mathcal{L}''_*$.

3.5. Proof of Theorem 1.5. $\nu \in \mathcal{M}$. Without loss of generality, we can assume that ν is continuous. Expand $\nu = \nu_1 + \nu_2 + \dots + \nu_\infty$. Set

$$\mathcal{L}^k = \mathcal{L}[\nu_1, \dots, \nu_k, \sum_{j=k+1}^{\infty} \nu_j, 0, 0, \dots; \nu_\infty], \quad \mathcal{L}^\infty := \mathcal{L}[\nu_1, \nu_2, \dots; \nu_\infty].$$

Let \mathcal{L}_*^k , \mathcal{L}_*^∞ be the same measure spaces with measures multiplied by t . Let $\text{id}^k : \mathcal{L}^k \rightarrow \mathcal{L}_*^k$, $\text{id}^\infty : \mathcal{L}^\infty \rightarrow \mathcal{L}_*^\infty$ denote the identical maps.

Iterating arguments of the two previous subsections, we obtain that the closure of $S[\nu, 0, \dots, 0]$ contains elements id^k for any finite k . Consider a map $\alpha_k : \mathcal{L}^k \rightarrow \mathcal{L}^\infty$ constructed in the following way. It is identical on $\mathbb{R}_1^\times, \dots, \mathbb{R}_k^\times$ send the semi-line \mathbb{R}_{k+1}^\times to $\prod_{j \geq k+1} \mathbb{R}_j$ preserving the measure. In the same way we construct a map $\beta_k : \mathcal{L}_*^k \rightarrow \mathcal{L}_*^\infty$. It is easy to see that the sequence

$$\chi_k := \beta_k \circ \text{id}_k \circ \alpha_k^{-1} : \mathcal{L}^\infty \rightarrow \mathcal{L}_*^\infty.$$

converges to id^∞ .

3.6. Construction of the function ψ . Here we obtain the continuous section $\mathcal{M} \rightarrow \text{Gms}$. Consider the distribution function $z = F(y)$ of the measure ν and the inverse function $y = G(z)$. If y_0 is a discontinuity point of F , we set $G(z) = y_0$ on the segment $[F(y_0 - 0), F(y_0) + y_0]$. If F takes some value z_0 on a segment of nonzero length, then $G(z_0)$ is not defined. Further, we set $\psi(x) = \int_0^x G(z) dz$.

⁵Recall that any two Lebesgue spaces with continuous probabilistic measures are equivalent, see e.g., [1].

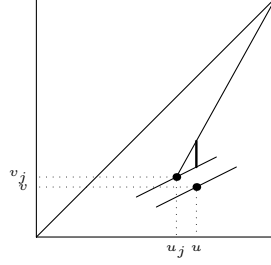


Figure 3: To proof of Proposition 1.7: $(u, v) = H(s)$, $(u_j, v_j) = H_j(s)$. We mark an interval of possible values of $\psi_j(u)$.

3.7. Proof of Proposition 1.7. Let ν_j converges to ν in \mathcal{M} , $y = \psi_j(x)$, $y = \psi(x)$ be the corresponding maps $[0, 1] \rightarrow [0, 1]$. We must prove that ψ_j converges to ψ in Gms.

1) Let $\nu \in \mathcal{M}$. Consider the map $\mathbb{R}^\times \rightarrow [0, 1] \times [0, 1]$ given by the formula

$$H : s \mapsto (\nu[(0, s)], (t \cdot \nu)[(0, s)]).$$

It is easy to see that we get the graph of the functions ψ , from which we remove all straight segments. The convergence $\nu_j \rightarrow \nu$ means the point-wise convergence of the maps $H_j(s) \rightarrow H(s)$. From this it is easy to derive that ψ_j converges to ψ point-wise (See Fig. 3). In virtue of monotonicity and continuity of our functions, the point-wise convergence implies the uniform convergence.

2) Let us show that derivatives ψ'_j converge ψ' a.s. Take a point a , where all derivatives $\psi'_j(a)$, $\psi'(a)$ are defined. Let ℓ_j , ℓ - be tangent lines to graphs of ψ_j , ψ at a . Suppose that $\psi'_j(a)$ does not converge to $\psi'(a)$. Choose a subsequence $\psi'_{n_k}(a)$ convergent to $\alpha \neq \psi'(a)$. Consider the limit line ℓ_{n_k} , i.e.,

$$\ell^\circ : y = \alpha(x - a) + \psi(a)$$

It is easy to see (for more details, see [14], Addendum, §6) that the graph $y = \psi(x)$ is located upper this line. I.e., ℓ° is the second supporting line at a (the first one was the tangent line), this contradicts to the existence of $\psi'(a)$.

3) Now we prove a weak convergence of operators $T_{1/2}(\psi_j) \rightarrow T_{1/2}(\psi)$ in $L^2[0, 1]$. Let f, h be continuous functions. We must check that the following expressions approach zero

$$\begin{aligned} \left| \int_0^1 f(\psi_j(x)) \psi'_j(x)^{1/2} h(x) dx - \int_0^1 f(\psi(x)) \psi'(x)^{1/2} h(x) dx \right| &\leq \\ &\leq \int_0^1 |f(\psi_j(x)) - f(\psi(x))| \psi'_j(x)^{1/2} h(x) dx + \end{aligned} \quad (3.4)$$

$$+ \int_0^1 |f(\psi(x)) (\psi'_j(x)^{1/2} - \psi'(x)^{1/2}) h(x)| dx \quad (3.5)$$

In (3.4) the convergence $f(\psi_j(x)) \rightarrow f(\psi(x))$ is uniform and

$$\int_0^1 \psi_j^{1/2}(x) \leq \int_0^1 (\psi_j^{1/2})^2(x) = 1.$$

By the Fatou Lemma, (3.4) tends to zero. Further notice that for functions $\psi \in \mathcal{G}$ we have a priory estimation

$$\psi'(x) \leq \frac{1 - \psi(x)}{1 - x} \leq \frac{1}{1 - x}.$$

Hence the convergence in the integral (3.5) is dominated on each segment $[0, 1 - \varepsilon]$. This implies that integrals $\int_0^{1-\varepsilon}(\dots)$ approach zero. Further, denote $C = (\max |f(x)| \cdot \max |g(x)|)$,

$$\begin{aligned} \int_{1-\varepsilon}^1 (\dots) &\leq C \int_{1-\varepsilon}^1 (\psi_j'(x)^{1/2} + \psi'(x)^{1/2}) dx \leq \varepsilon C \int_{1-\varepsilon}^1 (\psi_j'(x) + \psi'(x)) dx = \\ &= \varepsilon C [(1 - \psi_j(1 - \varepsilon)) + (1 - \psi(1 - \varepsilon))] \end{aligned}$$

and this value is small for small ε . \square

3.8. Proof of Theorem 1.8. Cut M into pieces $A_{ij} := M_i \cap g^{-1}M_j$, and also into pieces $B_{ij} = gM_{ij} = g(M_i) \cap M_j$. We get a collection of maps $A_{ij} \rightarrow B_{ij}$. Now the question is reduced to a canonical form of each map.

References

- [1] Rohlin, V. A. *On the fundamental ideas of measure theory*. Mat. Sbornik N.S. 25(67), (1949). 107–150. Amer. Math. Soc. Translation 1952, (1952). no. 71, 55 pp.
- [2] Neretin, Yu. A. *Spreading maps (polymorphisms), symmetries of Poisson processes, and matching summation*. J. Math. Sci. (N. Y.) 126 (2005), no. 2, 1077–1094
- [3] Neretin, Yu. *Symmetries of Gaussian measures and operator colligations*. J. Funct. Anal. 263 (2012), no. 3, 782–802.
- [4] Kechris, A. S. *Classical descriptive set theory*. Springer, New York, 1995.
- [5] Neretin, Yu.A. *Categories of bistochastic measures and representations of some infinite- dimensional groups*. Sbornik Math. 75, No.1, 197-219 (1993);
- [6] Pestov, V. *Dynamics of infinite-dimensional groups. The Ramsey–Dvoretzky–Milman phenomenon*. American Mathematical Society, Providence, RI, 2006.
- [7] Reed, M., Simon, B. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York-London, 1972.

- [8] Raikov, D. A. *On the completion of topological groups.* (Russian) Izvestia Akad. Nauk SSSR 10, (1946). 513-528.
- [9] Bourbaki, N. *Éléments de mathématique. Première partie. (Fascicule III.) Livre III; Topologie générale. Chap. 3: Groupes topologiques. Chap. 4: Nombres réels.* (French) Troisième édition revue et augmentée Actualités Sci. Indust., No. 1143. Hermann, Paris 1960.
- [10] Rokhlin, V. A. *Metric classification of measurable functions.* (Russian) Uspehi Mat. Nauk (N.S.) 12 (1957), no. 2(74), 169-174.
- [11] Fleming, R. J.; Jamison, J. E. *Isometries on Banach spaces: function spaces.* Chapman & Hall/CRC, Boca Raton, FL, 2003.
- [12] Köthe, G. *Topological vector spaces. I.* Springer, New York 1969.
- [13] Neretin Yu.A., *On the boundary of the group of transformations leaving a measure quasi-invariant* Sbornik: Mathematics(2013),204(8):1161
- [14] Aleksandrov, A. D. *Intrinsic Geometry of Convex Surfaces.* OGIZ, Moscow-Leningrad, 1948. English transl.: *Alexandrov selected works. Part II.* Edited by S. S. Kutateladze. Chapman & Hall/CRC, Boca Raton, FL, 2006.

Math.Dept., University of Vienna,
 Institute for Theoretical and Experimental Physics
 MechMath Dept., Moscow State University
 neretin(frog)mccme.ru
 URL: <http://www.mat.univie.ac.at/~neretin/>